SEPARATION OF RELATIVELY QUASICONVEX SUBGROUPS

JASON FOX MANNING AND EDUARDO MARTÍNEZ-PEDROZA

Abstract. Suppose that all hyperbolic groups are residually finite. The following statements follow: In relatively hyperbolic groups with peripheral structures consisting of finitely generated nilpotent subgroups, relatively quasiconvex subgroups are separable; Geometrically finite subgroups of non-uniform lattices in rank one symmetric spaces are separable; Kleinian groups are subgroup separable. We also show that LERF for finite volume hyperbolic 3–manifolds would follow from LERF for closed hyperbolic 3–manifolds.

The method is to reduce, via combination and filling theorems, the separability of a relatively quasiconvex subgroup of a relatively hyperbolic group G to the separability of a quasiconvex subgroup of a hyperbolic quotient $\bar{G}$. A result of Agol, Groves, and Manning is then applied.

1. Main Results

A subgroup H of a group G is called *separable* if for any $g \in G \setminus H$ there is a homomorphism π onto a finite group such that $\pi(g) \notin \pi(H)$. A group is called *residually finite* if the trivial subgroup is separable, and a group is called *subgroup separable* or *LERF* if every finitely generated subgroup is separable. For example, Hall showed that free groups are LERF in [15]. It follows from a theorem of Mal′cev [21] that polycyclic (and in particular finitely generated nilpotent) groups are LERF. A group is called *slender* if every subgroup is finitely generated. Polycyclic groups are also slender, by a result of Hirsch [16].

Given a relatively hyperbolic group with peripheral structure consisting of LERF and slender subgroups, we study separability of relatively quasiconvex subgroups. This is connected, via filling constructions, to residual finiteness of hyperbolic groups. It is not known whether all word-hyperbolic groups are residually finite. Consequences of a positive or negative answer to this question have been explored by several authors; see for example [2, 19, 20, 24, 27, 29]. In particular, the main result of [2] is the following.

Theorem 1.1. [2] *If all hyperbolic groups are residually finite, then every quasiconvex subgroup of a hyperbolic group is separable.*

We extend this result, answering a question in [2], as follows:

Theorem 1.2. *Suppose that all hyperbolic groups are residually finite. If G is a torsion free relatively hyperbolic group with peripheral structure consisting of subgroups which are LERF and slender, then any relatively quasiconvex subgroup of G is separable.*

This extension has some interesting corollaries. A *pinched Hadamard manifold* is a simply connected Riemannian manifold with pinched negative curvature. In [5], Bowditch gave several equivalent definitions of geometrical finiteness for discrete subgroups of the isometry group of a pinched Hadamard manifold, generalizing the

notion of geometrical finiteness in Kleinian groups. The next theorem summarizes some useful facts about these groups. (Statement (1) can be found in [13] or [6]; statement (2) follows from the Margulis lemma; and statement (3) is [18, Corollary 1.3].)

Theorem 1.3. [13, 18, 6] *Let X be a pinched Hadamard manifold and let G be a geometrically finite subgroup of $Isom(X)$.*

(1) *G is relatively hyperbolic, relative to a collection of representatives of conjugacy classes of maximal parabolic subgroups.*
(2) *Maximal parabolic subgroups of G are virtually nilpotent.*
(3) *A subgroup H of G is relatively quasiconvex if and only if H is geometrically finite.*

Rank one symmetric spaces are pinched Hadamard manifolds. We therefore have the following corollary of Theorem 1.2:

Corollary 1.4. *Suppose that all hyperbolic groups are residually finite. Let G be a discrete, geometrically finite subgroup of the isometry group of a rank one symmetric space. (For example, G could be a lattice.) All the geometrically finite subgroups of G are separable.*

In case the symmetric space is $\mathbb{H}^3$, more can be said (see Section 5 for the proof).

Corollary 1.5. *If all hyperbolic groups are residually finite, then all finitely generated Kleinian groups are LERF.*

Briefly, Theorem 1.2 is proved by combining one of Martínez-Pedroza's combination theorems in [22] with Theorem 1.1 and the Dehn filling technique of [14, 27]. We next give a more detailed discussion.

Definition 1.6. A relatively quasiconvex subgroup H of G is called *fully quasiconvex* if for any subgroup $P \in \mathcal{P}$ and any $f \in G$, either $H \cap P^f$ is finite or $H \cap P^f$ is a finite index subgroup of P^f. (Here $P^f = fPf^{-1}$.)

Using the work in [22], we show the following.

Theorem 1.7. *Let G be a group hyperbolic relative to a collection of slender and LERF subgroups. Suppose that Q is a relatively quasiconvex subgroup of G and g is an element of G not in Q. Then there exists a fully quasiconvex subgroup H which contains Q and does not contain g.*

Remark 1.8. In case G is a finite volume hyperbolic 3–manifold group and Q is the fundamental group of a quasi-fuchsian surface, Theorem 1.7 can be proved using geometric arguments like those in [10, 11]. Such geometric arguments were applied (in a different way) to separability questions in [3] (cf. [30]).

Using the work in [2] and Theorem 1.7, we prove the following separation result of relatively quasiconvex subgroups by maps onto word hyperbolic groups. (A part of this theorem can be interpreted as saying that G is "quasiconvex extended residually hyperbolic".)

Theorem 1.9. *Let G be a torsion free group hyperbolic relative to a collection of slender and LERF subgroups. For any relatively quasiconvex subgroup Q of G and any element $g \in G$ such that $g \notin Q$, there is a fully quasiconvex subgroup H of G, and a surjective homomorphism $\pi\colon G \longrightarrow \bar{G}$ such that*

(1) *$Q < H$,*
(2) *$\bar{G}$ is a word-hyperbolic group,*
(3) *$\pi(H)$ is a quasiconvex subgroup of $\bar{G}$,*
(4) *$\pi(g) \notin \pi(H)$.*

We can prove Theorem 1.2 from Theorem 1.9 as follows:

Proof of Theorem 1.2. Let $Q < G$ be relatively quasiconvex, and let $g \in G \setminus Q$. By Theorem 1.9 there is a fully quasiconvex $H < G$ containing Q but not g, and a quotient $\pi\colon G \to K$ so that $\pi(g) \notin \pi(H)$, K is hyperbolic, and $\pi(K)$ is quasiconvex.

Assuming all hyperbolic groups are residually finite, Theorem 1.1 implies that there is a finite group F and a quotient $\phi\colon K \to F$ so that $\phi(\pi(g)) \notin \phi(\pi(H))$. Since $\phi(\pi(H))$ contains $\phi(\pi(Q))$, the map $\phi \circ \pi$ serves to separate g from Q. □

Remark 1.10. The torsion-free hypotheses in Theorems 1.9 and 1.2 are not really necessary. We sketch the necessary changes to our argument in Appendix B. If one is primarily interested in Theorem 1.2 in the special case of virtually polycyclic peripheral subgroups, we have the following simple argument, pointed out to us by the referee:

Let G be a relatively hyperbolic group, relative to a collection $\mathcal{P} = \{P_1, \ldots, P_m\}$ of virtually polycyclic subgroups. An easy argument shows that each P_i contains a finite index normal subgroup which is torsion free. Moreover, G contains only finitely many finite order non-parabolic elements, up to conjugacy [26, Theorem 4.2]. It then follows from Osin's version of the relatively hyperbolic Dehn filling theorem that there is a filling $G \xrightarrow{\pi} G(N_1', \ldots, N_m')$, so that $G(N_1', \ldots, N_m')$ is hyperbolic, and no non-trivial torsion element of G is in the kernel of π. Assuming hyperbolic groups are residually finite, the group $G(N_1', \ldots, N_m')$ has a torsion-free subgroup S of finite index. The preimage $G_0 = \pi^{-1}(S)$ is a torsion-free finite index subgroup of G. Again under the assumption that hyperbolic groups are residually finite, Theorem 1.2 implies that G_0 is QCERF, and so (applying Corollary 2.2 below) G must also be QCERF.

The paper is organized as follows. In Section 2.5 we give a definition of relatively hyperbolic group which suits the purposes of this paper and is equivalent to the more standard definitions in the literature. In Section 3 we recall a combination theorem for relatively quasiconvex subgroups from [22] and prove Theorem 1.7. In Section 4 we recall some definitions and results on fillings of relatively hyperbolic groups and prove Theorem 1.9. In Section 5, we give two applications to separability questions on hyperbolic 3–manifolds: Corollary 1.5 and Proposition 5.3. In Appendix A, we prove a result we need on the equivalence of various definitions of relative quasiconvexity, and in Appendix B, we sketch how to prove our results in the presence of torsion.

2. Preliminaries

2.1. **Separability.** Let G be a group. Recall that the *profinite topology* on G is the smallest topology on G in which all finite index subgroups and their cosets are closed. The group G is residually finite if and only if this topology is Hausdorff. A subgroup H is separable if and only if it is a closed subset of G, with this topology.

Given a subgroup $G_0 < G$, one can ask whether the profinite topology on G_0 coincides with the topology induced by the profinite topology on G. In general, the

topologies are quite different, but in case G_0 is finite index, the topologies coincide. In particular, we have:

Lemma 2.2. *Let $G_0 < G$ be a finite index subgroup, and let $C_0 \subseteq G_0$. The following conditions are equivalent:*

(1) *C_0 is closed in the profinite topology on G_0.*
(2) *C_0 is closed in the profinite topology on G.*
(3) *$C_0 = C \cap G_0$ for some set C which is closed in the profinite topology on G.*

Proof. Suppose C_0 is closed in G_0. Since the finite index subgroups of G_0 and their cosets generate the topology on G_0, we can write C_0 as a finite union of arbitrary intersections of cosets

$$C_0 = \bigcup_{i=1}^{n} \bigcap_{i \in I_j} g_i K_i \tag{1}$$

where every K_i is a finite index subgroup of G_0. But since G_0 is finite index in G, each of the K_i appearing in equation (1) is also finite index in G. This C_0 is a finite union of intersections of closed sets in the profinite topology on G, so C_0 is closed in G. Thus condition (1) implies condition (2).

Trivially, condition (2) implies condition (3). To see that condition (3) implies condition (1), we first establish the following.

Claim 2.3. *If $g \in G$, and $K < G$ is finite index, then $gK \cap G_0$ is closed in the profinite topology on G_0.*

Proof. Let $K_0 = K \cap G_0$. Since K_0 is finite index in K, the subgroup K is a finite union of cosets $K = \cup_{i=1}^{p} g_i K_0$, and so $gK = \cup_{i=1}^{p} g g_i K_0$ is as well. Since a coset of K_0 in G either lies inside G_0 or in its complement, it follows that $gK \cap G_0$ is a finite union of cosets of K_0 in G_0. Since K_0 is finite index in G_0, these cosets are closed in the profinite topology on G_0 and therefore $gK \cap G_0$ is closed as well. □

Suppose then that $C_0 = C \cap G_0$ for C closed in G. We have $C = \bigcup_{i=1}^{n} \bigcap_{i \in I_j} g_i K_i$ where now the K_i are arbitrary finite index subgroups of G, and the g_i are arbitrary elements of G. We thus have

$$C_0 = C \cap G_0 = \bigcup_{i=1}^{n} \bigcap_{i \in I_j} (g_i K_i) \cap G_0. \tag{2}$$

By the claim, every $(g_i K_i) \cap G_0$ appearing in equation (2) is closed in G_0, and so C_0 is also closed in the profinite topology on G_0. □

Corollary 2.4. *Let $H < G$ be a pair of groups, and let G_0 be a finite index subgroup of G. Let $H_0 = H \cap G_0$. The subgroup H is separable in G if and only if H_0 is separable in G_0.*

Another immediate corollary is that LERFness and QCERFness are commensurability invariants.

2.5. **Relative hyperbolicity.** The notion of relative hyperbolicity has been studied by several authors with different equivalent definitions. The definition in this subsection is based on the work by D. Osin in [26]. Let G be a group, $\mathcal{P}$ denote a collection of subgroups $\{P_1, \ldots, P_m\}$, and S be a finite generating set which is assumed to be symmetric, i.e, $S = S^{-1}$. Denote by $\Gamma(G, \mathcal{P}, S)$ the Cayley graph

of G with respect to the generating set $S \cup \bigcup \mathcal{P}$. If p is a path between vertices in $\Gamma(G, \mathcal{P}, S)$, we will refer to its initial vertex as p_-, and its terminal vertex as p_+. The path p determines a word Label(p) in the alphabet $S \cup \bigcup \mathcal{P}$ which represents an element g so that $p_+ = p_- g$.

Definition 2.6 (Weak Relative Hyperbolicity)**.** The pair $(G, \mathcal{P})$ is *weakly relatively hyperbolic* if there is an integer $\delta \geq 0$ such that $\Gamma(G, \mathcal{P}, S)$ is δ–hyperbolic. We may also say that G is *weakly relatively hyperbolic, relative to* $\mathcal{P}$.

Definition 2.7 ([26])**.** Let q be a combinatorial path in the Cayley graph $\Gamma(G, \mathcal{P}, S)$. Sub-paths of q with at least one edge are called *non-trivial.* For $P_i \in \mathcal{P}$, a P_i–*component* of q is a maximal non-trivial sub-path s of q with $Label(s)$ a word in the alphabet P_i. When we don't need to specify the index i, we will refer to P_i–components as $\mathcal{P}$*–components.*

Two $\mathcal{P}$–components s_1, s_2 are *connected* if the vertices of s_1 and s_2 belong to the same left coset of P_i for some i. A $\mathcal{P}$–component s of q is *isolated* if it is not connected to a different $\mathcal{P}$–component of q. The path q is *without backtracking* if every $\mathcal{P}$–component of q is isolated.

A vertex v of q is called *phase* if it is not an interior vertex of a $\mathcal{P}$–component s of q. Let p and q be paths between vertices in $\Gamma(G, \mathcal{P}, S)$. The paths p and q are k*–similar* if

$$\max\{dist_S(p_-, q_-), dist_S(p_+, q_+)\} \leq k,$$

where $dist_S$ is the metric induced by the finite generating set S (as opposed to the metric in $\Gamma(G, \mathcal{P}, S)$).

Remark 2.8. A geodesic path q in $\Gamma(G, \mathcal{P}, S)$ is without backtracking, all $\mathcal{P}$–components of q consist of a single edge, and all vertices of q are phase.

Definition 2.9 (Bounded Coset Penetration (BCP))**.** The pair $(G, \mathcal{P})$ satisfies the *BCP property* if for any $\lambda \geq 1$, $c \geq 0$, $k \geq 0$, there exists an integer $\epsilon(\lambda, c, k) > 0$ such that for p and q any two k–similar (λ, c)–quasi-geodesics in $\Gamma(G, \mathcal{P}, S)$ without backtracking, the following conditions hold:

(i.) The sets of phase vertices of p and q are contained in the closed $\epsilon(\lambda, c, k)$–neighborhoods of each other, with respect to the metric $dist_S$.
(ii.) If s is any $\mathcal{P}$–component of p such that $dist_S(s_-, s_+) > \epsilon(\lambda, c, k)$, then there exists a $\mathcal{P}$–component t of q which is connected to s.
(iii.) If s and t are connected $\mathcal{P}$–components of p and q respectively, then

$$\max\{dist_S(s_-, t_-), dist_S(s_+, t_+)\} \leq \epsilon(\lambda, c, k).$$

Remark 2.10. Our definition of the BCP property corresponds to the conclusion of Theorem 3.23 in [26].

Definition 2.11 (Relative Hyperbolicity)**.** The pair $(G, \mathcal{P})$ is *relatively hyperbolic* if the group G is weakly relatively hyperbolic relative to $\mathcal{P}$ and the pair $(G, \mathcal{P})$ satisfies the Bounded Coset Penetration property. If $(G, \mathcal{P})$ is relatively hyperbolic then we say G is *relatively hyperbolic, relative to* $\mathcal{P}$; if there is no ambiguity, we just say that G is relatively hyperbolic.

Remark 2.12. Definition 2.11 given here is equivalent to Osin's [26, Definition 2.35] for finitely generated groups: To see that Osin's definition implies 2.11, apply [26, Theorems 3.23]; to see that 2.11 implies Osin's definition, apply [26, Lemma

7.9 and Theorem 7.10]. For the equivalence of Osin's definition and the various other definitions of relative hyperbolicity see [18] and the references therein.

The definition of relative hyperbolicity is independent of finite generating set S.

3. Combination of Parabolic and quasiconvex Subgroups

In this section, G will be relatively hyperbolic, relative to a finite collection of subgroups $\mathcal{P}$, and S will be a finite generating set for G. Denote by $\Gamma(G, \mathcal{P}, S)$ the Cayley graph of G with respect to the generating set $S \cup \bigcup \mathcal{P}$.

3.1. **Parabolic and Quasiconvex Subgroups.**

Definition 3.2. The *peripheral* subgroups of G are the elements of $\mathcal{P}$. A subgroup of G is called *parabolic* if it can be conjugated into a peripheral subgroup.

Proposition 3.3. [26, Proposition 2.36] *The following conditions hold.*

1. *For any g_1, $g_2 \in G$, the intersection $P_i^{g_1} \cap P_j^{g_2}$ is finite unless $i = j$.*
2. *The intersection $P_i^g \cap P_i$ is finite for any $g \notin P_i$.*

In particular, if Q is a subgroup of G, then any infinite maximal parabolic subgroup of Q is of the form $Q \cap P_i^f$ for some $f \in Q$ and $P_i \in \mathcal{P}$.

Definition 3.4. [26, Definition 4.9] A subgroup Q of G is called *quasiconvex relative to $\mathcal{P}$* (or simply *relatively quasiconvex* when the collection $\mathcal{P}$ is fixed) if there exists a constant $\sigma \geq 0$ such that the following holds: Let f, g be two elements of Q, and p an arbitrary geodesic path from f to g in the Cayley graph $\Gamma(G, \mathcal{P}, S)$. For any vertex $v \in p$, there exists a vertex $w \in Q$ such that $dist_S(v, w) \leq \sigma$, where $dist_S$ is the word metric induced by S.

Remark 3.5. For more on different definitions of relative quasiconvexity in the literature, see Appendix A.

Theorem 3.6. [18, Theorem 9.1] *Let Q be a finitely generated relatively quasiconvex subgroup of G. The number of infinite maximal parabolic subgroups of Q up to conjugacy in Q is finite. Furthermore, if $\mathcal{O}$ is a set of representatives of these conjugacy classes, then Q is relatively hyperbolic, relative to $\mathcal{O}$.*

Remark 3.7. In [18], an extended definition of relative hyperbolicity is used which includes some countable but non-finitely generated groups. Using this extended definition, the assumption of finite generation in Theorem 3.6 is superfluous.

We note that in case all the peripheral subgroups are slender, relatively quasiconvex subgroups are necessarily finitely generated (see [18, Corollary 9.2]).

3.8. **Combination of Quasiconvex Subgroups.** For $g \in G$, $|g|_S$ denotes the distance from g to the identity element in the word metric induced by S.

Theorem 3.9. [22, Theorem 1.1] *Let Q be a relatively quasiconvex subgroup of G, and let P be a maximal parabolic subgroup of G. Suppose that $P^f = P_i$ for some $P_i \in \mathcal{P}$ and $f \in G$.*

There are constants $C = C(Q, P) \geq 0$ and $c = c(Q, P) \geq 0$ with the following property. Suppose $D \geq C$ and R is a subgroup of P such that

- *$P \cap Q < R$, and*
- *$|g|_S > D$ for any element $g \in R \setminus Q$.*

It follows that:

(1) *The subgroup $H = \langle Q \cup R \rangle$ is relatively quasiconvex and the natural map $Q *_{Q\cap R} R \longrightarrow H$ is an isomorphism.*
(2) *Every parabolic subgroup of H is conjugate in H to a parabolic subgroup of Q or R.*
(3) *For any $g \in H$, either $g \in Q$, or any geodesic from 1 to g in the relative Cayley graph $\Gamma(G, \mathcal{P}, S)$ has at least one P_i-component t such that $|t|_S > D - c$.*

Proof. Conclusions (1) and (2) rephrase [22, Theorem 1.1]. The proof of conclusion (3) is divided into two cases: $P \in \mathcal{P}$ and $P \notin \mathcal{P}$.

Case 1. $P \in \mathcal{P}$.

We summarize part of the argument for [22, Theorem 1.1] for conclusions (1) and (2); we then explain how conclusion (3) follows in this case.

Let $g \in Q *_{Q\cap R} R \setminus Q$. The element g has a *normal form*

$$g = g_1 h_1 \dots g_k h_k \tag{3}$$

where $g_j \in Q \setminus Q \cap R$ for $1 < j \leq k$, $h_j \in R \setminus Q \cap R$ for $1 \leq j < k$, either $g_1 = 1$ or $g_1 \in Q \setminus Q \cap R$, and either $h_k = 1$ or $h_k \in R \setminus Q \cap R$. We use the normal form to produce a path o in $\Gamma(G, \mathcal{P}, S)$ from 1 to the image of g by the natural map $Q *_{Q\cap R} R \longrightarrow H$ as follows. For each j between 1 and k, let u_j be a geodesic path in $\Gamma(G, \mathcal{P}, S)$ from $g_1 h_1 \cdots h_j$ to $g_1 h_1 \cdots h_j g_j$ (so that Label(u_j) represents g_j). Similarly, let v_j be a geodesic path from $g_1 h_1 \cdots g_{j-1}$ to $g_1 h_1 \cdots g_{j-1} h_j$ (so that Label(v_j) represents h_j). A path o from 1 to g in $\Gamma(G, \mathcal{P}, S)$ is given by

$$o = u_1 v_1 \dots u_k v_k.$$

(See Figure 1.)

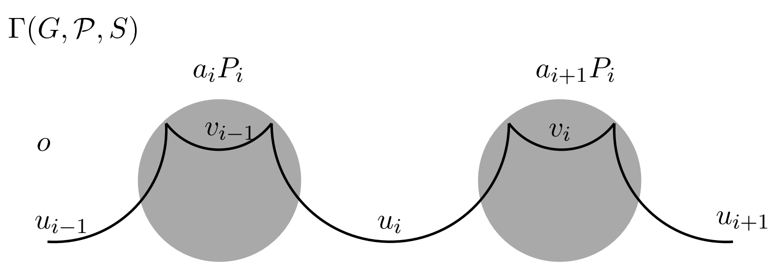

FIGURE 1. Part of the polygonal path o in $\Gamma(G, \mathcal{P}, S)$. For each i, the P_i–component of o containing the subsegment v_j is long with respect to the S–metric. This implies that the path o is a quasi-geodesic with different end-points.

Each subsegment v_j is part of a $\mathcal{P}$-component t_j of the path o. Let D be as in the hypothesis of the theorem. The penultimate inequality in the proof of Claim 2 in the proof of Lemma 5.1 of [22] is

$$|t_j|_S = dist_S((t_j)_-, (t_j)_+) > D - 2M(P, Q, \sigma), \tag{4}$$

where σ is the quasiconvexity constant for Q, and $M(P, Q, \sigma)$ is the constant provided by [22, Lemma 4.2] for the subgroups Q, P, and the constant σ. Let η be

the constant from Proposition 3.1 of [22]. If $D - 2M(P, Q, \sigma) > \eta$, then o is a $(\lambda, 0)$–quasi-geodesic with distinct endpoints. Let $C = \eta + 2M(P, Q, \sigma)$.

It follows from the argument just sketched that if $D > C$, then the natural map $Q *_{Q\cap R} R \longrightarrow H$ is an isomorphism. It can further be shown that H is a relatively quasiconvex subgroup and that the parabolic subgroups of H are conjugate into Q or R by elements of H (See [22, Lemmas 5.2 and 5.3]for details.). In other words, parts (1) and (2) hold for $P \in \mathcal{P}$ and C as above.

If $g \in H \setminus Q$ and p is a geodesic from 1 to g in $\Gamma(G, \mathcal{P}, S)$, then the $(\lambda, 0)$–quasi-geodesic o and the geodesic p are 0–similar. Since o has a $\mathcal{P}$–component of S–length at least $D - 2M(P, Q, \sigma)$, the Bounded Coset Penetration property (Definition 2.9) implies that p has a $\mathcal{P}$–component of S–length at least $D - 2M(P, Q, \sigma) - 2\epsilon(0, \lambda, 0)$. We have verified (3) of the Theorem for c equal to

$$c(Q, P) = 2M(P, Q, \sigma) + 2\epsilon(0, \lambda, 0)$$

in the special case that $f = 1$ and $P \in \mathcal{P}$.

Case 2. $P \notin \mathcal{P}$, but $P^f \in \mathcal{P}$ for some $f \in G \setminus P$.

Since $P^f \in \mathcal{P}$ and Q^f is relatively quasiconvex, by [22, Theorem 1.1] and Case 1, all three conclusions of Theorem 3.9 hold for Q^f and P^f and some constants $C' = C(Q^f, P^f) > 0$ and $c' = c(Q^f, P^f) > 0$. Define

$$C = C' + 2|f|_S + 3\epsilon(1, 0, |f|_S),$$

and

$$c = c' + 2|f|_S + 2\epsilon(1, 0, |f|_S),$$

where $\epsilon(1, 0, |f|_S)$ is the constant of Definition 2.9 on the Bounded Coset Penetration property. Now we show that the theorem holds for the subgroups P and Q, and the constants C and c. Let R be a subgroup of Q satisfying the hypothesis of the theorem for a constant $D > C$.

If $r \in R \setminus Q$, then $|r|_S \geq D$, by hypothesis. It follows that

$$|r^f|_S \geq D - 2|f|_S \geq C'.$$

We therefore have:

(1) The subgroup $H^f = \langle Q^f \cup R^f \rangle$ is relatively quasiconvex and the natural map $Q^f *_{(Q\cap R)^f} R^f \longrightarrow H^f$ is an isomorphism. Since relative quasiconvexity is preserved by conjugation, $H = \langle Q \cup R \rangle$ is relatively quasiconvex. Obviously the map $Q *_{Q\cap R} R \longrightarrow H$ is also an isomorphism. In other words, conclusion (1) holds for Q and P and the constant C.

(2) Every parabolic subgroup of H^f is conjugate in H^f to a parabolic subgroup of Q^f or R^f. Parabolicity is preserved under conjugation, so the same property (conclusion (2) of the theorem) holds for the subgroups Q, R, and H, and the constant C.

(3) For any $h \in H^f$, either $h \in Q^f$, or any geodesic from 1 to h in the relative Cayley graph $\Gamma(G, \mathcal{P}, S)$ has at least one P_i–component t such that $|t|_S > D - 2|f|_S - c'$.

It remains to see why conclusion (3) of the Theorem holds with the chosen constant c. Let $g \in H \setminus Q$ and let p be a geodesic from 1 to g in $\Gamma(G, \mathcal{P}, S)$. We must show that p has a P_i–component of S–length at least $D - c$. Let q be a geodesic from 1 to fgf^{-1}. Since fgf^{-1} belongs to $H^f \setminus Q^f$, the geodesic q has a P_i–component u of S–length at least $D - 2|f|_S - c'$. Let r be the geodesic starting

at f, with the same label as p. Thus r joins f to fg, and the geodesics q and r are $|f|_S$–similar (see Figure 2). By the Bounded Coset Penetration property, r has a

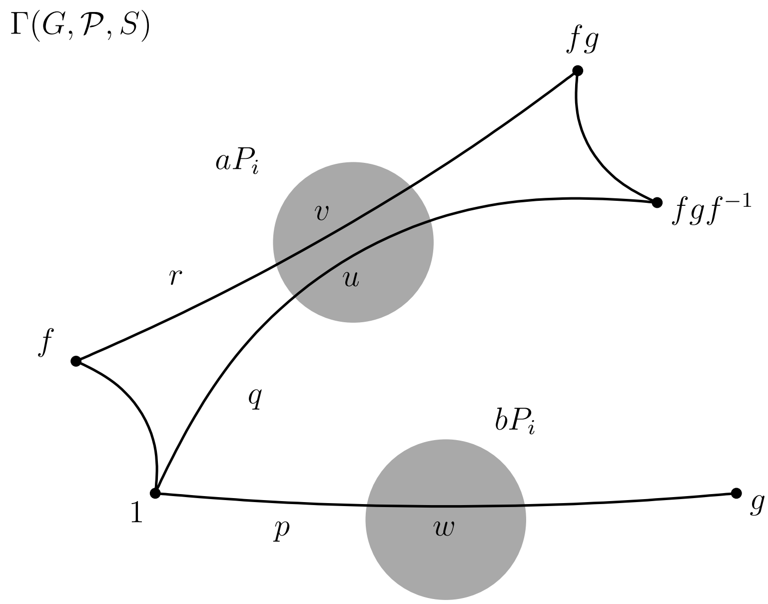

FIGURE 2. The geodesics q and r are $|f|_S$–similar and q contains a large $\mathcal{P}$–component u. By the BCP-property, r has a large $\mathcal{P}$–component v. Since p and r have the same word-label, p has a large $\mathcal{P}$–component w.

P_i–component v of S–length at least

$$D - 2|f|_S - c' - 2\epsilon(1, 0, |f|_S) = D - c.$$

Since $p = f^{-1}r$ and r have the same labels, it follows that p has a P_i–component w of S–length at least $D - c$.

□

Corollary 3.10. [22, Lemma 5.4] *Suppose that G, Q, P, and R are as in the hypothesis of Theorem 3.9 and that $Q \cap P$ is a proper subgroup of R.*

If $\{K_1, \ldots, K_n\}$ is a collection of representatives of the maximal infinite parabolic subgroups of Q up to conjugacy in Q so that $K_1 = P \cap Q$, then $\{R, K_2, \ldots, K_n\}$ is a collection of representatives of the maximal parabolic subgroups of H up to conjugacy in H.

Proof. By Theorem 3.9 (2), a maximal parabolic subgroup of H is conjugate to R or K_i for some $i = 2, \ldots n$. Hence $\{R, K_2, \ldots, K_n\}$ is a collection of representatives of maximal parabolic subgroups. That all these subgroups are different up to conjugacy follows from the algebraic structure of H as an amalgamated product. In particular, since K_i and K_j are not conjugate in Q, they are not conjugate in $Q *_{Q\cap R} R$. The subgroup $R < Q *_{Q \cap R} R$ is not conjugate to a subgroup of Q since $Q \cap R$ is a proper subgroup of R. □

Proof of Theorem 1.7. Suppose that every subgroup in $\mathcal{P}$ is LERF and slender. Let Q be a relatively quasiconvex subgroup of G and let g be an element of G not in Q. Let $\{K_1, \ldots, K_n\}$ be a collection of representatives of maximal infinite parabolic subgroups of Q up to conjugacy in Q; such a collection exists by Theorem 3.6. By Proposition 3.3, for each K_i there is a peripheral subgroup $P_i \in \mathcal{P}$ and $f_i \in G$ such that $K_i < P_i^{f_i}$.

We will construct an ascending sequence of relatively quasiconvex subgroups

$$Q = Q_0 < Q_1 < \cdots < Q_n = H$$

such that for each $k \in \{1, \ldots, n\}$ the following properties hold.

(1) For $0 \leq j \leq k$, the subgroup $Q_k \cap P_j^{f_j}$ is finite index in $P_j^{f_j}$.
(2) $\{Q_k \cap P_1^{f_1}, \ldots, Q_k \cap P_k^{f_k}, K_{k+1}, \ldots, K_n\}$ is a collection of representatives of the maximal parabolic subgroups of Q_k up to conjugation in Q_k.
(3) $g \notin Q_k$.

It will follow that $H = Q_n$ is a fully quasiconvex subgroup which contains Q and does not contain the element g.

Choose a geodesic p from 1 to g in the relative Cayley graph $\Gamma(G, \mathcal{P}, S)$. Let L be an upper bound for the S–length of the P_k–components of the path p.

We now show how to construct Q_k, assuming that Q_{k-1} has already been constructed. Let C and c be the constants provided by Theorem 3.9 for the subgroups Q_{k-1} and $P_k^{f_k}$. Since $P_k^{f_k}$ is slender, $K_k = Q_{k-1} \cap P_k^{f_k}$ is finitely generated. Define a finite set $F \subset P_k^{f_k} \setminus K_k$ by

$$F = \begin{cases} \{p \in P_k^{f_k} \setminus K_k \mid |p|_S \leq L + C + c\} \cup \{g\} & \text{if } g \in P_k^{f_k} \\ \{p \in P_k^{f_k} \setminus K_k \mid |p|_S \leq L + C + c\} & \text{otherwise.} \end{cases}$$

Because $P_k^{f_k} \cong P_k$ is LERF, we may find a finite index subgroup R_k of $P_k^{f_k}$ satisfying

- $K_k < R_k$, and
- $f \notin R_k$ for all $f \in F$.

In particular, $|h|_S > L + C + c$ for any $h \in R_k \setminus Q_{k-1}$. Let $Q_k = \langle Q_{k-1} \cup R_k \rangle$. Note that the hypotheses of the combination Theorem 3.9 (and hence those of Corollary 3.10) are satisfied for the relatively quasiconvex subgroup Q_{k-1} and the parabolic subgroup R_k.

We now verify properties (1)–(3) for the subgroup Q_k just constructed. Property (1) follows from the fact that Q_k contains $Q_j \cap P_j^{f_j}$ for each j between 1 and $k-1$, and also contains R_k. Corollary 3.10 implies that property (2) is satisfied. By Theorem 3.9(3) any geodesic in the Cayley graph $\Gamma(G, S \cup \bigcup \mathcal{P})$ from 1 to an element of element of Q_k which is not in $Q_{k-1} \cup R_k$ has a P_k–component of S–length greater than $L + C$; it follows that the element g does not belong to Q_k, and so property (3) is also satisfied. This concludes the construction of the group Q_k, and the theorem follows by taking $H = Q_n$. □

4. Fillings of Relatively Hyperbolic Groups

Let G be torsion free and relatively hyperbolic, relative to a collection of subgroups $\mathcal{P} = \{P_1, \ldots, P_m\}$, and let $S \subset G$ be a finite generating set of G. Suppose that $S \cap P_i$ is a generating set of P_i for each i.

Definition 4.1. A *filling* of G is determined by a collection of subgroups $\{N_i\}_{i=1}^m$ such that for each i, N_i is a normal subgroup of P_i; these subgroups are called *filling kernels*. The quotient of G by the normal subgroup generated by $\bigcup_{i=1}^m N_i$ is denoted by $G(N_1, \dots, N_m)$.

The following result is due (in the present setting) independently to Groves and Manning [14, Theorem 7.2 and Corollary 9.7] and to Osin [27, Theorem 1.1]. (Osin actually proves a more general result, in which G may have torsion.)

Theorem 4.2. [14, 27] *Let F be a finite subset of G. There exists a constant B depending on G, $\mathcal{P}$, S, and F with the following property. If a collection of filling kernels $\{N_i\}_{i=1}^m$ satisfies $|f|_S > B$ for every nontrivial $f \in \cup_i N_i$, then*

1. *the natural map $\imath_i : P_i/N_i \longrightarrow G(N_1, \dots, N_m)$ is injective,*
2. *$G(N_1, \dots, N_m)$ is relatively hyperbolic relative to $\{\imath_i(P_i/N_i)\}_{i=1}^m$, and*
3. *the projection $G \longrightarrow G(N_1, \dots, N_m)$ is injective on F.*

4.3. **Fillings and Quasiconvex Subgroups.** Let H be a relatively quasiconvex subgroup of G. A filling $G \longrightarrow G(N_1, \dots, N_m)$ is an *H–filling* if whenever $H \cap P_i^g$ is non-trivial, $N_i^g \subset P_i^g \cap H$.

Theorem 4.4. [2, Propositions 4.3 and 4.5] *Let $H < G$ be a finitely generated relatively quasiconvex subgroup and $g \in G \setminus H$. There is a finite subset $F \subset G$ depending on H and g with the following property.*

If $\pi\colon G \longrightarrow G(N_1, \dots, N_m)$ is an H–filling which is injective on F, then $\pi(H)$ is a relatively quasiconvex subgroup of $G(N_1, \dots, N_m)$, and $\pi(g) \notin \pi(H)$.

Remark 4.5. Propositions 4.3 and 4.5 in [2] use a different definition of relative quasiconvexity than the one we use here. In particular, their definition requires the subgroup to be finitely generated. We show in Appendix A (specifically Corollary A.11), that the definition used in [2] is equivalent to the one we use here under the assumption that the subgroup is finitely generated.

Proof of Theorem 1.9. Suppose that every subgroup in $\mathcal{P}$ is LERF and slender. Let Q be a relatively quasiconvex subgroup of G and let g be an element of G not in Q. By Theorem 1.7, there is a fully quasiconvex subgroup H which contains Q and does not contain g. We must choose filling kernels $\{N_i\}_{i=1}^m$ such that $\pi\colon G \longrightarrow G(N_1, \dots, N_m)$ is an H–filling, $G(N_1, \dots, N_m)$ is a word hyperbolic group, $\pi(H)$ is a quasiconvex subgroup, and $\pi(g) \notin \pi(H)$.

By Theorem 3.6, there is a collection $\{K_1, \dots, K_n\}$ of representatives of the infinite maximal parabolic subgroups of H up to conjugacy in H. For each $r \in \{1, \dots, n\}$ there is an integer $i_r \in \{1, \dots, m\}$ and an element $f_r \in G$ such that $K_r^{f_r}$ is a finite index subgroup of P_{i_r}. The index i_r is determined by r, but there may be many distinct f_r with this property. On the other hand, there will be only finitely many conjugates ${K_i}^g$ in P_{i_r}. Let I_r be the intersection of these conjugates; the group I_r is a finite index normal subgroup of P_{i_r}. For $i \in \{1, \dots, m\}$ define the subgroup M_i of P_i as

$$M_i = \begin{cases} P_i & \text{if } \{r \mid i_r = i\} = \emptyset \\ \bigcap \{I_r \mid i_r = i\} & \text{if } \{r \mid i_r = i\} \neq \emptyset \ . \end{cases}$$

Put another way, if some conjugate of H intersects P_i nontrivially,

$$M_i = \bigcap \{{K_r}^g \mid {K_r}^g \cap P_i \neq \{1\},\ r \in \{1, \dots, n\}, \text{ and } g \in G\}.$$

In other words, M_i is the intersection of P_i with all the ${K_r}^g$ which lie in P_i. Because M_i is a finite intersection of finite index normal subgroups of P_i, M_i is a finite index normal subgroup of P_i. From the definition, $M_i < {K_r}^g$ whenever $r \in \{r \mid i_r = i\}$ and ${K_r}^g < P_i$.

By Theorem 4.4, there is a finite subset $F \subset G$ such that if $\pi\colon G \longrightarrow \bar{G}$ is any H–filling which is injective on F, then $\pi(g) \notin \pi(H)$ and $\pi(H)$ is relatively quasiconvex. Let $B > 0$ be the constant from Theorem 4.2, applied to this finite subset F.

Since each P_i is residually finite, there is some finite index $\hat{N}_i \lhd P_i$ so that $|p|_S > B$ for all $p \in \hat{N}_i \setminus \{1\}$. Let $N_i = \hat{N}_i \cap M_i$. Since N_i is an intersection of two finite index normal subgroups of P_i, N_i is a finite index normal subgroup of P_i. Moreover, $|p|_S > B$ for every $p \in N_i \setminus \{1\}$.

Lemma 4.6. *$G \to G(N_1, \ldots, N_m)$ is an H–filling.*

Proof. Let $P_i \in \mathcal{P}$, $f \in G$, and suppose that $H \cap {P_i}^f \neq \{1\}$. We must show that ${N_i}^f < H$. Obviously it suffices to show that ${M_i}^f < H$. Since G is torsion free, $H \cap {P_i}^f$ is infinite and $H \cap {P_i}^f = {K_r}^h$ for some $h \in H$. The group

$$ {M_i}^f = \bigcap \{ {K_r}^g \mid {K_r}^g \cap {P_i}^f \neq \{1\},\ r \in \{1, \ldots, n\}, \text{ and } g \in G \} $$

must therefore be contained in $H \cap {P_i}^f$, and thus in H. □

We now claim the subgroup $H < G$ and the filling $G \to \bar{G} := G(N_1, \ldots, N_m)$ satisfy the conclusions of Theorem 1.9. Indeed, conclusion (1) of Theorem 1.9 is satisfied by construction. By Theorem 4.2.(2), the quotient $\bar{G}$ is hyperbolic relative to a collection of finite groups, hence $\bar{G}$ is hyperbolic. Conclusion (2) is thus established. According to Lemma 4.6, $G \to \bar{G}$ is an H–filling. By Theorem 4.2.(3), $G \to \bar{G}$ is injective on F. Since the peripheral subgroups are slender, H is finitely generated [18, Corollary 9.2]. We therefore may apply Theorem 4.4 to obtain conclusions (3) and (4) of Theorem 1.9. Having established all the conclusions, we have proved the theorem. □

5. Applications to 3–manifolds

5.1. **Kleinian groups.** Recall that a *Kleinian group* is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, the group of isometries of hyperbolic 3–space. In this subsection we show that if all hyperbolic groups are residually finite, then all finitely generated Kleinian groups are LERF.

Proof of Corollary 1.5. By Selberg's lemma [4], every finitely generated Kleinian group contains a torsion-free subgroup of finite-index. Applying Corollary 2.4 it suffices to consider torsion-free Kleinian groups. Let G be a torsion-free Kleinian group and let $H < G$ be a finitely generated subgroup. There are two cases:

Case 1. $\mathbb{H}^3/G$ has finite volume.

Either H is geometrically finite or not. Suppose first that H is not geometrically finite. By a result of Canary [9, Corollary 8.3] together with the positive solution to the Tameness Conjecture [1, 8], H must be a *virtual fiber subgroup* of G. This implies that there is a finite index subgroup of $G_0 < G$ whose intersection H_0 with H is normal in G_0, and so that $G_0/H_0 \cong \mathbb{Z}$. The group H_0 is obviously separable in G_0, so H is separable in G by Corollary 2.4.

If H is geometrically finite, then it is relatively quasiconvex, by Theorem 1.3, and we may apply Theorem 1.2.

Case 2. $\mathbb{H}^3/G$ has infinite volume.

In this case it follows from the Scott core theorem and Thurston's geometrization theorem for Haken manifolds that G is isomorphic to a geometrically finite Kleinian group G' (see, for example, [23, Theorem 4.10]). If H' is the image of H in G' then obviously H is separable in G if and only if H' is separable in G'. We therefore may as well assume that G is geometrically finite to begin with.

Since G is geometrically finite and infinite covolume, every finitely generated subgroup of G is geometrically finite by an argument of Thurston (see [25, Proposition 7.1] or [23, Theorem 3.11] for a proof). In particular, H is geometrically finite, so Theorem 1.3 implies that H is a relatively quasiconvex subgroup of G. To finish, we apply Theorem 1.2 again. □

5.2. **Subgroup separability in finite volume 3–manifolds.** Here we prove that if compact hyperbolic 3–manifold groups are QCERF then finite volume hyperbolic 3–manifold groups are LERF.

Proposition 5.3. *If all fundamental groups of compact hyperbolic 3–manifolds are QCERF, then all fundamental groups of finite volume hyperbolic 3–orbifolds are LERF.*

Proof. Let $G = \pi_1(M)$, where M is some finite volume hyperbolic 3–orbifold. Applying Corollary 2.4, we may pass to a finite cover and assume that M is an orientable manifold. It follows that G is relatively hyperbolic, relative to some finite collection $\mathcal{P} = \{P_i, \dots, P_m\}$ of subgroups, each of which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Now suppose $Q < G$ is some finitely generated subgroup. We must show that Q is separable. If Q is geometrically infinite, then we argue as we did in the proof of 1.5 that Q is the fundamental group of a virtual fiber, and thus separable. We may thus suppose that Q is geometrically finite, and therefore a relatively quasiconvex subgroup of G.

Let $g \in G \setminus Q$. We then apply Theorem 1.7 to enlarge Q to a fully quasiconvex subgroup H not containing g. Let $F \subset G$ be the finite set obtained by applying Theorem 4.4 to G, H, and g. Let B_1 be the constant from Theorem 4.2 applied to G and F, with respect to some generating set S for G. We will choose a cyclic filling kernel $N_i < P_i$ for each $P_i \in \mathcal{P}$. The Hyperbolic Dehn Surgery Theorem of Thurston [28, 17] implies there is some constant B_2 so that if the generators of the N_i are chosen to have length greater that B_2, then the filling $G(N_1, \dots, N_m)$ will be the (orbifold) fundamental group of a hyperbolic orbifold obtained by attaching orbifold solid tori to the boundary components of a compact core of M. Let $B = \max\{B_1, B_2\}$.

For each $P_i \in \mathcal{P}$ we choose some cyclic $N_i < P_i$. For each i let $n_i \in P_i$ satisfy $|n_i|_S > B$. There are at most finitely many conjugates ${P_i}^{t_1}, \dots {P_i}^{t_k}$ so that ${P_i}^{t_j} \cap H$ is nonempty; for each such j, the group ${P_i}^{t_j} \cap H$ is finite index in ${P_i}^{t_j}$. If there are no such conjugates, we choose $N_i = \langle n_i \rangle$. Otherwise, we let $N_i = \langle n_i^\alpha \rangle$, where the power $\alpha \in \mathbb{N}$ is chosen so that $n_i^\alpha \in H^{{t_j}^{-1}}$ for each j, and so $|n_i^\alpha|_S > B$.

With the $N_i < P_i$ chosen as above, the H–filling $G(N_1, \dots, N_m)$ is a compact hyperbolic 3–orbifold group. Let $\pi\colon G \to G(N_1, \dots, N_m)$ be the quotient map. By Theorem 4.4, $\pi(H)$ is a quasiconvex subgroup of $G(N_1, \dots, N_m)$, not containing

$\pi(g)$. By assumption, compact hyperbolic 3–manifold groups (and thus compact hyperbolic 3–orbifold groups) are QCERF. There is therefore some finite group F, and some $\phi\colon G(N_1,\ldots,N_m) \to F$ with $\phi(\pi(g)) \notin \phi(\pi(H))$. Since $\phi(\pi(H))$ contains $\phi(\pi(Q))$, we have separated g from Q in a finite quotient. □

Remark 5.4. No part of the proof of Proposition 5.3 rests in an essential way on the results in this paper or in [2], but only on facts about hyperbolic 3–manifolds and 3–orbifolds which could be deduced by geometric arguments, based on the Gromov-Thurston 2π Theorem. On the other hand, Proposition 5.3 is a nice illustration of the general principle that a relatively hyperbolic group which can be approximated by QCERF Dehn fillings must itself be QCERF.

6. Acknowledgments

Thanks very much to Ian Agol and Daniel Groves, both for useful conversations, and for helpful comments on an earlier draft of this note. We also thank the referee for insightful comments and corrections.

Manning was partially supported by NSF grant DMS-0804369. Martínez-Pedroza was funded as a Britton Postdoctoral Fellow at McMaster University, grant A4000.

Appendix A. On the equivalence of definitions of quasiconvexity

The current paper relies heavily on results about relatively quasiconvex subgroups of relatively hyperbolic groups proved in the papers [22] and [2]. These papers unfortunately use different definitions of relative quasiconvexity, but we show in this appendix that the two definitions agree, at least for finitely generated subgroups.

First, a few words about the literature. Dahmani [12] and Osin [26] studied classes of subgroups of relatively hyperbolic groups which they called *relatively quasiconvex*, intending to generalize the notion of quasiconvexity in hyperbolic groups. Dahmani's definition was a dynamical one, whereas Osin's was Definition 3.4; Osin's definition was used in [22]. Hruska in [18] gave several definitions of relative quasiconvexity in the setting of countable (not necessarily finitely generated) relatively hyperbolic groups, including definitions based on Osin's and Dahmani's, and showed that they are equivalent. The authors of [2] were mainly interested in relatively hyperbolic structures on groups which were already hyperbolic, and used a definition of relative quasiconvexity (based more closely on the usual metric notion of quasiconvexity) different from any of those in [18]. The definition in [2] applies only to a finitely generated subgroup of a finitely generated relatively hyperbolic group.

Throughout this section G will be relatively hyperbolic, relative to a finite collection of subgroups $\mathcal{P} = \{P_1,\ldots,P_n\}$, and S will be a finite generating set for G. The *cusped space* (recalled below) for G with respect to $\mathcal{P}$ and S will be denoted by $X(G,\mathcal{P},S)$, and $d(\cdot,\cdot)$ will denote the path metric on the cusped space. In particular, we will not need the word metric on G, but only the metric induced by this path metric. For more detailed definitions and background on cusped spaces for relatively hyperbolic groups we refer the reader to [14] and [18]; we sketch the construction and recall some terminology here for the reader's convenience.

Let A be a discrete metric space with metric ρ. The *combinatorial horoball based on A* is a graph $\mathcal{H}(A)$ with vertex set $A \times \mathbb{Z}_{\geq 0}$, so that

- (a,n) is connected by an edge to $(a,n+1)$ for any $a \in A$, and

- for $n \geq 1$, (a,n) is connected to (a',n) whenever $\rho(a,a') \leq 2^n$.

Edges of the first type are called *vertical*; edges of the second type are called *horizontal*. We say that a vertex (a,n) of $\mathcal{H}(A)$ has *depth* n. If A is a subset of a path metric space Y, we may *attach a horoball to* Y *along* A by gluing $A \subseteq Y$ to $A \times \{0\} \subset \mathcal{H}(A)$, and taking the obvious path metric on the union. If G is finitely generated by S, we take Y to be the Cayley graph of G with respect to S; any subset of G inherits a discrete metric from the path metric on the Cayley graph. The *cusped space* $X(G,\mathcal{P},S)$ is the space obtained by simultaneously attaching horoballs to Y along all left cosets tP for $P \in \mathcal{P}$. A vertex v of $X(G,\mathcal{P},S)$ corresponds either to a group element, if it lies in the Cayley graph of G, or otherwise to a triple (tP,g,n) where tP is a left coset of an element of $\mathcal{P}$, the element g lies in tP, and $n > 0$ is the depth of v in the attached horoball $\mathcal{H}(tP)$. In what follows we do not distinguish between v and the corresponding group element or triple.

The group G is relatively hyperbolic, relative to $\mathcal{P}$, if and only if the space $X(G,\mathcal{P},S)$ is Gromov hyperbolic (see for example [14, Theorem 3.25]). If so, then G acts on $X(G,\mathcal{P},S)$ *geometrically finitely*,[1] meaning in particular:

(1) Given any $n \geq 0$, let X_n be the subset of $X(G,\mathcal{P},S)$ obtained by deleting all vertices of height greater than n. Then G acts cocompactly on X_n (which is an example of what Hruska calls a *truncated space* for the action of G on $X(G,\mathcal{P},S)$).
(2) For fixed n, there are only finitely many components of of $X(G,\mathcal{P},S) \setminus X_n$, up to the action of G. (The components of $X(G,\mathcal{P},S) \setminus X_{n-1}$ are called *n–horoballs*, for $n \geq 1$. If n is understood, we call them *horoballs*. A 0–horoball is a 1–neighborhood of a 1–horoball, and is equal to $\mathcal{H}(tP)$ for some coset tP of some $P \in \mathcal{P}$.)

Relatively Quasiconvex Subgroups according to Agol–Groves–Manning. Suppose that H is a relatively hyperbolic group, and let $\mathcal{D} = \{D_1, \ldots, D_m\}$ be the peripheral subgroups of H and T a finite generating set for H. Let $\phi\colon H \to G$ be a homomorphism. If every $\phi(D_i) \in \mathcal{D}$ is conjugate in G into some $P_j \in \mathcal{P}$, we say that the map ϕ *respects the peripheral structure on* H.

Given such a map ϕ, one can extend it to a map $\check{\phi}$ between zero-skeletons of cusped spaces in the following way: For each $D_i \in \mathcal{D}$, choose an element $c_i \in G$ (of minimal length) and some P_{j_i} such that $\phi(D_i) \subseteq cP_{j_i}c^{-1}$. For $h \in H$, $\check{\phi}(h) = \phi(h)$. For a vertex (sD_i, h, n) in a horoball of $X(H,\mathcal{D},T)$, define

$$\check{\phi}(sD_i, h, n) = (\phi(s)c_iP_{j_i}, \phi(h)c_i, n).$$

Lemma A.1. [2, Lemma 3.1] *Let $\phi\colon H \to G$ be a homomorphism which respects the peripheral structure on H. The extension $\check{\phi}$ defined above is H–equivariant and lipschitz. If ϕ is injective, then $\check{\phi}$ is proper.*

Recall that, for $C \geq 0$, a subset A of a geodesic metric space X is *C–quasiconvex* if every geodesic with endpoints in A lies in a C–neighborhood of A. The subset is *quasiconvex* if it is C–quasiconvex for some C. The following is a slight paraphrase of the definition from [2]:

Definition A.2. (QC-AGM) [2, Definition 3.11] Let G be as above, and let $H < G$ be finitely generated by a set T. We say that H is *(QC-AGM) relatively quasiconvex* in $(G,\mathcal{P})$ if, for some finite collection of subgroups $\mathcal{D}$ of H,

[1] Hruska uses the term "cofinitely."

(1) H is relatively hyperbolic, relative to $\mathcal{D}$, and
(2) if $\iota\colon H \to G$ is the inclusion, then the map $\breve{\iota}\colon X(H,\mathcal{D},T)^0 \to X(G,\mathcal{P},S)^0$ described in Lemma A.1 has quasiconvex image.

Relatively Quasiconvex Subgroups according to Hruska. The following definition is direct from Hruska [18], where it is called *QC-3*. Hruska shows in [18] that this definition is equivalent to several others, including our Definition 3.4.

Definition A.3. (QC-H)[18, Definition 6.6] A subgroup $H \leq G$ is *(QC-H) relatively quasiconvex* if the following holds. Let (X,ρ) be some (any) proper Gromov hyperbolic space on which $(G,\mathcal{P})$ acts geometrically finitely. Let $X - U$ be some (any) truncated space for G acting on X. For some (any) basepoint $x \in X - U$ there is a constant $\mu \geq 0$ such that whenever c is a geodesic in X with endpoints in the orbit Hx, we have

$$c \cap (X - U) \subseteq \mathcal{N}_\mu(Hx),$$

where the neighborhood is taken with respect to the metric ρ on X.

Remark A.4. The meaning of "some (any)" in Definition A.3 just means that the word "some" can be replaced by "any" without affecting which subgroups of G are (QC-H) relatively quasiconvex. Thus "Definition" A.3 has some non-definitional content, established in [18, Proposition 7.5 and 7.6].)

Definition A.5. Let $A \subset X = X(G,\mathcal{P},S)$ be a horoball, and let $R > 0$. We say that a geodesic γ *penetrates the horoball A to depth R* if there is a point $p \in \gamma \cap A$ at distance at least R from $X \setminus A$. We say that *A is R–penetrated by the subgroup H* if there is a geodesic γ with endpoints in H penetrates the horoball A to depth R.

The goal of this subsection is to prove the following proposition.

Proposition A.6. *Let $H < G$ be (QC-H) relatively quasiconvex. Then there is a constant $R = R(G,\mathcal{P},S,H)$, so that whenever a 0–horoball is R–penetrated by H, the intersection of H with the stabilizer of that horoball is infinite.*

Before the proof, we quote a proposition from [18] and prove two lemmas.

Proposition A.7. [18, Proposition 9.4] *Let G have a proper, left invariant metric d, and suppose xH and yK are arbitrary left cosets of subgroups of G. For each constant L there is a constant $L' = L'(G,d,xH,yK)$ so that in the metric space (G,d) we have*

$$\mathcal{N}_L(xH) \cap \mathcal{N}_L(yK) \subseteq \mathcal{N}_{L'}(xHx^{-1} \cap yKy^{-1}).$$

Lemma A.8. *Let H be a (QC-H) relatively quasiconvex subgroup of G. Let A be a 0–horoball of $X(G,\mathcal{P},S)$, whose stabilizer is P^t for $P \in \mathcal{P}$. If A is R–penetrated by H for all $R > 0$, then $H \cap P^t$ is infinite.*

Proof. It suffices to show that, for every $M > 0$, there is some h in $H \cap P^t$ with $d(1,h) > M$.

Let μ be the quasiconvexity constant of Definition A.3 for H and the space X' which consists of all vertices in $X(G,\mathcal{P},S)$ at depth 0. Let C be the constant given by Proposition A.7 such that

$$\mathcal{N}_\mu(H) \cap tP \subseteq \mathcal{N}_C(H \cap tPt^{-1}),$$

where the neighborhoods are taken in the cusped space.

Suppose that γ is a geodesic with endpoints in H which penetrates the horoball A to depth $M+C$. The first and last points of $\gamma \cap A$ are group elements, a and b, both in the coset tP. Since H is (QC-H) relatively quasiconvex, a and b are elements of $\mathcal{N}_\mu(H) \cap tP$ and therefore (using Proposition A.7) there are elements h_1 and h_2 in $H \cap P^t$ such that $d(h_1, a) \le C$ and $d(h_2, b) \le C$. Since $d(a,b) \ge 2(M+C)$,

$$d(1, h_1^{-1}h_2) = d(h_1, h_2) \ge 2(M+C) - 2C \ge 2M > M. \qquad \square$$

Lemma A.9. *Let H be a (QC-H) relatively quasiconvex subgroup of G. Let μ be the quasiconvexity constant of Definition A.3 for H and the space $X' = X_0$ which is obtained from $X(G, \mathcal{P}, S)$ by deleting all vertices of positive depth.*

Let $R > 0$, and let A be a 0–horoball, stabilized by P^t for $P \in \mathcal{P}$. If A is R–penetrated by H, then there is a horoball A' so that

1. *$A' = hA$ for some $h \in H$,*
2. *$d(A', 1) \le \mu$, and*
3. *A' is R–penetrated by H.*

Proof. Suppose that γ is a geodesic with endpoints h_1 and h_2 in H which penetrates the horoball A to depth R. Let a and b be the first and last vertices of $\gamma \cap A$. By (QC-H) relative quasiconvexity, there is some $h \in H$ so that $d(a,h) \le \mu$. The geodesic $h^{-1}\gamma$ goes between $h^{-1}h_2$ and $h^{-1}h_2$, and penetrates the horoball $A' = h^{-1}A$ to depth R. Moreover,

$$d(1, A') \le d(1, h^{-1}a) = d(a, h) \le \mu. \qquad \square$$

Proof of Proposition A.6. Suppose there is no such number R. There must be a sequence of integers $R_i \to \infty$ and a sequence of 0–horoballs $\{A_i\}$, so that, for each i, the horoball A_i is R_i–penetrated by H, but the intersection of the stabilizer of A_i with H is finite.

For $h \in H$, the stabilizer of hA_i is conjugate (by h) to the stabilizer of A_i. Using Lemma A.9, we can therefore assume that $d(1, A_i) \le \mu$ for each i. By passing to a subsequence, we can therefore assume that the sequence $\{A_i\}$ is constant. It follows that A_0 is R_i–penetrated by H for all i. Lemma A.8 then implies that the intersection of H with the stabilizer of A_0 is infinite, which is a contradiction. $\square$

Equivalence of the two definitions. In this section, G will be a relatively hyperbolic group, relative to a finite collection of subgroups $\mathcal{P}$, and S will be a finite generating set for G. Let $X(G, \mathcal{P}, S)$ be the cusped space for G with respect to $\mathcal{P}$ and S, and let δ be its hyperbolicity constant.

Theorem A.10. *Let H be a finitely generated subgroup of G. Then H is (QC-H) relatively quasiconvex if and only if H is (QC-AGM) relatively quasiconvex.*

Proof. One direction is easy. Suppose that $H < G$ is (QC-AGM) relatively quasiconvex, generated by the finite set T, and with peripheral subgroups $\mathcal{D}$. Recall that to define $\breve{\imath}\colon X(H, \mathcal{D}, T)^0 \longrightarrow X(G, \mathcal{P}, S)^0$, an element $c_i \in G$ was chosen for each $D_i \in \mathcal{D}$ so that $D_i \subset c_i P_{j_i} c_i^{-1}$. Let $C = \max\{d(1, c_i) \mid D_i \in \mathcal{D}\}$, and let C_q be the constant of quasiconvexity in the definition of (QC-AGM) quasiconvexity. As remarked at the beginning of the Appendix, the cusped space $X = X(G, \mathcal{P}, S)$ is acted on geometrically finitely by G, and the subspace $X - U = X_0 \subset X(G, \mathcal{P}, S)$ obtained by deleting 1–horoballs is a truncated space for the action. Moreover, as explained in Remark A.4, it suffices to find a μ which works for this choice of X

and $X - U$, and for the H–orbit of 1 in X. Let $x, y \in H$, and let γ be any geodesic joining them in X. Let z be a vertex of γ contained in X_0. By (QC-AGM), there is some point $w \in \breve{\iota}(X(H, \mathcal{D}, T)^0)$ so that $d(z, w) \leq C_q$. It follows that $w \in X_{C_q}$, but any point in $\breve{\iota}(H, \mathcal{D}, T) \cap X_{C_q}$ is at most $C + C_q$ away from some point in H. It follows that z is no further than $\mu := C + 2C_q$ from H, and so H is (QC-H) relatively quasiconvex.

We now establish the other direction. Let H be a subgroup of G, and suppose that H is (QC-H) relatively quasiconvex. Let $\mathcal{D}$ be a collection of representatives of the H–conjugacy classes of infinite maximal parabolic subgroups of H. By [18, Theorem 9.1], H is relatively hyperbolic, relative to $\mathcal{D}$. By Lemma A.1, the inclusion $\iota\colon H \longrightarrow G$ extends to a lipschitz map of (0–skeletons of) cusped spaces

$$\breve{\iota}\colon X(H, \mathcal{D}, T)^0 \longrightarrow X(G, \mathcal{P}, S)^0.$$

We need to prove that the image $Y = \breve{\iota}(X(H, \mathcal{D}, T)^0)$ of $\breve{\iota}$ is quasiconvex.

Let R be the constant provided by Proposition A.6 for the subgroup H. Let $X' = X_{100\delta+R}$ be the subspace of $X(G, \mathcal{P}, S)$ consisting of all vertices at depth at most $100\delta + R$. Since H is (QC-H) relatively quasiconvex, there is a constant μ such that for any geodesic ζ in $X(G, \mathcal{P}, S)$ with endpoints in H,

$$\zeta \cap X' \subset \mathcal{N}_\mu(H) \subset \mathcal{N}_\mu(Y),$$

where the neighborhoods are taken with respect to the metric on $X(G, \mathcal{P}, S)$.

Let x and y be vertices of Y and let γ be a geodesic between them. We will show that the vertices of γ are contained in the M-neighborhood of Y, where M is a constant independent of x, y, and γ. We divide the proof into five (not necessarily disjoint) cases.

Case 1. The points x and y lie deeper than 10δ in the same horoball.

By recalling some easily verified properties of the geometry of horoballs, we will show that γ is contained in the M_1-neighborhood of Y, where

$$M_1 = 6.$$

To begin with, the 10δ–horoball containing x and y is convex (see [14, Lemma 3.26]). Second, any geodesic with the same endpoints as γ is Hausdorff distance at most 4 from γ. Finally, there is a geodesic γ' of a particularly nice form with the same endpoints as γ. The geodesic γ' is a *regular geodesic*, which means that all its edges are vertical, except for at most three consecutive horizontal edges at maximum depth (see [14, Lemma 3.10]). Since the vertical subsegments of γ' start at points in Y and are vertical, they stay in Y, and so γ' stays in a 2–neighborhood of Y. As γ is contained in a 4–neighborhood of γ', we have γ contained in a 6–neighborhood of Y.

Case 2. The points x and y are elements of H, they are in the neighborhood of radius μ of a horoball $\mathcal{H}(tP)$, and the geodesic γ penetrates the horoball $\mathcal{H}(tP)$ to depth larger than $100\delta + R$.

In this case, we will approximate γ by a regular geodesic inside $\mathcal{H}(tP)$ with (possibly different) endpoints in Y. Without loss of generality, assume that x is the identity, and so $d(1, t) \leq \mu$.

By Proposition A.6, the intersection $H \cap P^t$ is infinite. It follows that $H \cap P^t = D^s$ for some $D \in \mathcal{D}$ and $s \in H$. We claim s can be chosen so that $d(1, s) < K$ for a

constant K independent of x, y, and γ. Indeed, we observe that the set

$$W = \{(r,P) \in G \times \mathcal{P} \mid d(1,r) \leq \mu, \#(H \cap P^r) = \infty\}$$

is finite. For each $w = (r,P) \in W$ choose $u_w \in H$ so that $H \cap P^r = D^u$ for some $D \in \mathcal{D}$; we let K be the maximum of $d(1,u_w)$ over all $w \in W$.

We further claim that there is an element $y' \in H \cap P^t$ such that $d(y,y') \leq L$, for a constant L independent of x, y, and γ. Indeed, for each $w = (r,P)$ in the set W defined above, Proposition A.7 implies we can find an $L_w > 0$ so that

$$H \cap \mathcal{N}_\mu(rP) \subseteq \mathcal{N}_{L_w}(H \cap P^r);$$

we let L be the maximum L_w over all $w \in W$.

Recall that to define $\breve{\imath}\colon\ X(H,\mathcal{D},T)^0 \longrightarrow X(G,\mathcal{P},S)^0$, an element $c_i \in G$ was chosen for each $D_i \in \mathcal{D}$ so that $D_i \subset c_i P_{j_i} c_i^{-1}$. Let

$$C = \max\{d(1,c_i) \mid D_i \in \mathcal{D}\}. \tag{5}$$

The subgroup D is equal to D_i for some i, and we set $c = c_i$ for the same i.

Consider the elements $(sD, s, 10\delta)$ and $(sD, y's, 10\delta)$ of $X(H,\mathcal{D},T)$ and their corresponding images in Y given by $(scP, sc, 10\delta)$ and $(scP, y'sc, 10\delta)$. The points $(scP, sc, 10\delta)$ and $(scP, y'sc, 10\delta)$ belong to the same 10δ–horoball, which is convex in $X(G,\mathcal{P},S)$, as we noted in Case 1. Also as noted in Case 1, there is a regular geodesic γ' joining the points $(scP, sc, 10\delta)$ and $(scP, y'sc, 10\delta)$; since the endpoints lie in Y, the geodesic γ' is contained in the 2–neighborhood of Y.

On the other hand, the endpoints of the geodesics γ and γ' are close, namely,

$$d(1,(scP,sc,10\delta)) \leq d(1,s) + d(1,c) + 10\delta \leq K + C + 10\delta,$$

and

$$d(y,(scP,y'sc,10\delta)) \leq d(y,y') + d(y',(scP,y'sc,10\delta)) \leq L + K + C + 10\delta.$$

Since $X(G,\mathcal{P},S)$ is δ–hyperbolic, the Hausdorff distance between γ' and γ is at most the distance between endpoints plus 2δ. Thus if

$$M_2 = 2\delta + K + L + C + 10\delta,$$

then γ is contained in the M_2-neighborhood of Y. This completes this case.

Case 3. Suppose x and y are elements of H.

We split γ into subsegments $\gamma_1, \gamma_2, \dots, \gamma_k$ such that no γ_i contains any group element (depth 0 vertex) in its interior, but the endpoints of each γ_i are group elements. Observe that each γ_i is either a single edge or a geodesic segment contained in a 0–horoball. Furthermore, since H is (QC-H) relatively quasiconvex, the endpoints of each γ_i are contained in the μ-neighborhood of H. We claim that each γ_i is contained in the M_3-neighborhood of Y, where

$$M_3 = 110\delta + R + 3\mu + 2 + M_2.$$

If γ_i is an edge, then the claim is immediate, so we suppose γ_i is contained in a 0–horoball $\mathcal{A}$. First, suppose γ_i does not penetrate $\mathcal{A}$ to depth $110\delta + R + 2\mu$. An easy argument shows that the length of a geodesic in a combinatorial horoball is at most twice its maximum depth plus 4, so we have $|\gamma_i| < 220\delta + 2R + 2\mu + 4$, and γ_i is therefore contained in the $(110\delta + R + 3\mu + 2)$–neighborhood of H. In particular, γ_i is contained in M_3–neighborhood of Y.

Suppose on the other hand that γ_i penetrates the horoball $\mathcal{A}$ to depth $110\delta + R + 2\mu$. Let h_1 and h_2 be elements of H which are at distance at most μ from

the endpoints of γ_i, and let α be a geodesic between them. Since $X(G,\mathcal{P},S)$ is δ–hyperbolic, the Hausdorff distance between γ_i and α is at most $2\delta+\mu$. It follows that α penetrates the horoball $\mathcal{A}$ to depth $100\delta+R+\mu$, and hence it satisfies the condition of Case 2. Therefore, γ_i is in the $(2\delta+\mu+M_2)$–neighborhood (and hence in the M_3–neighborhood) of Y.

Case 4. Suppose x and y lie at depth no more than 50δ in $X(G,\mathcal{P},S)$.

If $x \in Y$ lies in a 1–horoball, then $x=(tP,hc_i,n)$ for some $P\in\mathcal{P}$, some $h\in H$, some $i\in\{1,\dots,m\}$, and some $n\le 50\delta$; otherwise, $x\in H$. In any case, there is an element $h_1\in H$ such that $d(x,h_1)\le 50\delta+C$, where C is the constant defined in (5). By the same argument, there is an element $h_2\in H$ such that $d(x,h_2)\le 50\delta+C$. Since $X(G,\mathcal{P},S)$ is δ–hyperbolic, the Hausdorff distance between γ and any geodesic γ' between h_1 and h_2 is at most $52\delta+C$. We may apply Case 3 to γ', and deduce that γ is contained in the M_4–neighborhood of Y, where

$$M_4 = 52\delta + C + M_3.$$

Case 5. Suppose either x or y lies inside a 50δ–horoball, but we are not in Case 1.

Here we follow the proof of the last case of [2, Proposition 3.12]. If x or y lies in a horoball, it is connected by a vertical path to a point in the right coset Hc_i at depth 0 in $X(G,\mathcal{P},S)$. It is therefore possible to modify γ (by appending and deleting (mostly) vertical paths lying in a 3–neighborhood of Y) to a 10δ–local geodesic γ' with endpoints within C of H; the geodesic γ is contained in a 3–neighborhood of $\gamma'\cup Y$. By [7, III.H.1.13(3)], γ' is a $(\frac{7}{3},2)$–quasi-geodesic. Since quasi-geodesics track geodesics, there is a constant L_Q depending only on δ and C, and a geodesic γ'' with endpoints in H such that the Hausdorff distance between γ' and γ'' is at most L_Q. By Case 3, γ'' is contained in the M_2-neighborhood of Y. Let

$$M_5 = 3 + L_Q + M_2,$$

and observe that γ is contained in the M_5-neighborhood of Y.

Finally, we set $M=\max\{M_1,\dots,M_5\}$, and note that M does not depend on the vertices x and y of Y, or on the geodesic γ joining them. It follows that $Y=\breve{\iota}(X(H,\mathcal{D},T)^0)$ is M–quasiconvex in $X(H,\mathcal{D},T)$, and so H is (QC-AGM) relatively quasiconvex in $(G,\mathcal{P})$. □

Applying the main result of Hruska [18] on the equivalence of various definitions of relative quasiconvexity (our Definition 3.4 is Hruska's (QC-5), and our Definition A.3 (QC-H) is Hruska's (QC-3)), we obtain the following useful fact.

Corollary A.11. *Let G be relatively hyperbolic, relative to $\mathcal{P}$, and let H be a finitely generated subgroup of G. The following are equivalent:*

(1) *H is a relatively quasiconvex subgroup of G, in the sense of Definition 3.4.*
(2) *H is a relatively quasiconvex subgroup of G, in the sense of Definition A.2.*

Appendix B. On extending the main result in the presence of torsion

In this section, we give some idea of the changes necessary to prove Theorem 1.9 (and therefore Theorem 1.2) in the presence of torsion. In this section, G is a relatively hyperbolic group, hyperbolic relative to a finite collection $\mathcal{P}$ of LERF and slender subgroups, and H is some relatively quasiconvex subgroup of G.

The main difference is that we must deal with the possibility that our relatively quasiconvex subgroup has finite but non-trivial maximal parabolic subgroups. Since a finite subgroup of a relatively hyperbolic group, may intersect arbitrary collections of parabolic subgroups, we have to ignore these intersections. This is already handled in the arguments of Section 3 by only amalgamating with parabolic subgroups which have infinite intersection with H to obtain the fully quasiconvex subgroup Q.

In Section 4, it is necessary to modify the definition of H–filling as follows:

Definition B.1. (Alternate definition in the presence of torsion.) Let H be a relatively quasiconvex subgroup of G. A filling $G \longrightarrow G(N_1, \ldots, N_m)$ is an H–*filling* if whenever $H \cap P_i^g$ is infinite, $N_i^g \subset P_i^g \cap H$.

With the new definition, we must check that the results from [2, Section 4.2] still hold. (We do not know how to prove the result about height from Section 4.3 of [2] in this more general setting, but we do not need it for our argument.) Examining the proofs from [2], the reader may check that it suffices to extend the technical [2, Lemma 4.2].

We sketch how to do so briefly, for the experts: In [2, Lemma 4.2], the hypothesis of an H–filling is used to deduce the existence of a nontrivial element of H which is also in a conjugate of a filling kernel fixing a certain horoball from the fact that a geodesic between elements of H penetrates that horoball deeply. The heart of the argument is showing that if the geodesic penetrates the horoball deeply, the intersection of H with the horoball stabilizer is infinite. In the torsion-free setting, it suffices to show that the intersection is nontrivial. The proof in the presence of torsion is given in the previous appendix as A.6. With this proposition, one can prove the extended version of [2, Lemma 4.2] in a straightforward manner, choosing slightly different constants to take the constant R from Proposition A.6 into account.

The proofs of Propositions 4.3 and 4.5 of [2] go through in exactly the same way, and we obtain the same statement as Theorem 4.4 above, but with the new meaning of H–filling. Using Osin's Dehn filling result in place of Theorem 4.2, the rest of the proof of Theorem 1.9 goes through as written, with the exception that each mention of a condition of the form "$A \cap B \neq \{1\}$" for A and B subgroups of G should be replaced by "$A \cap B$ is infinite".

References

[1] I. Agol. Tameness of hyperbolic 3-manifolds. Preprint at `arXiv:math/0405568v1`, 2004.

[2] I. Agol, D. Groves, and J. Manning. Residual finiteness, QCERF, and fillings of hyperbolic groups. *Geom. Topol.*, 13:1043–1073, 2009.

[3] I. Agol, D. D. Long, and A. W. Reid. The Bianchi groups are separable on geometrically finite subgroups. *Ann. of Math. (2)*, 153(3):599–621, 2001.

[4] Roger C. Alperin. An elementary account of Selberg's lemma. *Enseign. Math. (2)*, 33(3-4):269–273, 1987.

[5] B. H. Bowditch. Geometrical finiteness with variable negative curvature. *Duke Math. J.*, 77(1):229–274, 1995.

[6] B.H. Bowditch. Relatively hyperbolic groups. Preprint at `http://www.warwick.ac.uk/~masgak/papers/`, 1999.

[7] M. R. Bridson and A. Haefliger. *Metric Spaces of Non–Positive Curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften*. Springer–Verlag, Berlin, 1999.

[8] Danny Calegari and David Gabai. Shrinkwrapping and the taming of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 19(2):385–446 (electronic), 2006.

[9] Richard D. Canary. A covering theorem for hyperbolic 3-manifolds and its applications. *Topology*, 35(3):751–778, 1996.
[10] D. Cooper and D. D. Long. Virtually Haken Dehn-filling. *J. Differential Geom.*, 52(1):173–187, 1999.
[11] D. Cooper and D. D. Long. Some surface subgroups survive surgery. *Geom. Topol.*, 5:347–367 (electronic), 2001.
[12] François Dahmani. Combination of convergence groups. *Geom. Topol.*, 7:933–963 (electronic), 2003.
[13] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.
[14] D. Groves and J.F. Manning. Dehn filling in relatively hyperbolic groups. *Israel Journal of Mathematics*, 168:317–429, 2008.
[15] Marshall Hall, Jr. Coset representations in free groups. *Trans. Amer. Math. Soc.*, 67:421–432, 1949.
[16] K. A. Hirsch. On infinite soluble groups. I. *Proc. London Math. Soc. (2)*, 44:53–60, 1937.
[17] Craig D. Hodgson and Steven P. Kerckhoff. Universal bounds for hyperbolic Dehn surgery. *Ann. of Math. (2)*, 162(1):367–421, 2005.
[18] G.C. Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. Preprint at `arXiv:0801.4596`, 2008.
[19] Ilya Kapovich and Daniel T. Wise. The equivalence of some residual properties of word-hyperbolic groups. *J. Algebra*, 223(2):562–583, 2000.
[20] D. D. Long and G. A. Niblo. Subgroup separability and 3-manifold groups. *Math. Z.*, 207(2):209–215, 1991.
[21] A. I. Mal′cev. On homomorphisms onto finite groups. *American Mathematical Society Translations (2)*, 119:67–79, 1983.
[22] E. Martínez-Pedroza. Combination of quasiconvex subgroups of relatively hyperbolic groups. *Groups, Geometry and Dynamics*, 3:317–342, 2009.
[23] Katsuhiko Matsuzaki and Masahiko Taniguchi. *Hyperbolic manifolds and Kleinian groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1998. Oxford Science Publications.
[24] Ashot Minasyan. Separable subsets of GFERF negatively curved groups. *J. Algebra*, 304(2):1090–1100, 2006.
[25] John W. Morgan. On Thurston's uniformization theorem for three-dimensional manifolds. In *The Smith conjecture (New York, 1979)*, volume 112 of *Pure Appl. Math.*, pages 37–125. Academic Press, Orlando, FL, 1984.
[26] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.
[27] Denis V. Osin. Peripheral fillings of relatively hyperbolic groups. *Invent. Math.*, 167(2):295–326, 2007.
[28] William P. Thurston. Geometry and topology of three-manifolds. Princeton lecture notes available at `http://www.msri.org/publications/books/gt3m/`, 1980.
[29] Daniel T. Wise. The residual finiteness of negatively curved polygons of finite groups. *Invent. Math.*, 149(3):579–617, 2002.
[30] Daniel T. Wise. Subgroup separability of the figure 8 knot group. *Topology*, 45(3):421–463, 2006.

Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260-2900
E-mail address: `j399m@buffalo.edu`

Department of Mathematics & Statistics, McMaster University, Hamilton, ON, L8S 4K1, Canada.
E-mail address: `emartinez@math.mcmaster.ca`